

# HARMONIC COCYCLES, VON NEUMANN ALGEBRAS, AND IRREDUCIBLE AFFINE ISOMETRIC ACTIONS

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**ABSTRACT.** Let  $G$  be a compactly generated locally compact group and  $(\pi, \mathcal{H})$  a unitary representation of  $G$ . The 1-cocycles with coefficients in  $\pi$  which are harmonic (with respect to a suitable probability measure on  $G$ ) represent classes in the first reduced cohomology  $\bar{H}^1(G, \pi)$ . We show that harmonic 1-cocycles are characterized inside their reduced cohomology class by the fact that they span a minimal closed subspace of  $\mathcal{H}$ . In particular, the affine isometric action given by a harmonic cocycle  $b$  is irreducible (in the sense that  $\mathcal{H}$  contains no non-empty, proper closed invariant affine subspace) if the linear span of  $b(G)$  is dense in  $\mathcal{H}$ . The converse statement is true, if  $\pi$  moreover has no almost invariant vectors. Our approach exploits the natural structure of the space of harmonic 1-cocycles with coefficients in  $\pi$  as a Hilbert module over the von Neumann algebra  $\pi(G)'$ , which is the commutant of  $\pi(G)$ . Using operator algebras techniques, such as the von Neumann dimension, we give a necessary and sufficient condition for a factorial representation  $\pi$  without almost invariant vectors to admit an irreducible affine action with  $\pi$  as linear part.

## 1. INTRODUCTION

Let  $G$  be a locally compact group and  $(\pi, \mathcal{H})$  a continuous unitary (or orthogonal) representation of  $G$  on a complex (or real) Hilbert space  $\mathcal{H}$ . Recall that a 1-*cocycle* with coefficients in  $\pi$  is a continuous map  $b : G \rightarrow \mathcal{H}$  such that  $b(gh) = b(g) + \pi(g)b(h)$  for all  $g, h \in G$  and that a 1-cocycle is a *coboundary* if it is of the form  $\partial_v$  for some  $v \in \mathcal{H}$ , where  $\partial_v(g) = \pi(g)v - v$  for  $g \in G$ . The space  $Z^1(G, \pi)$  of 1-cocycles with coefficients in  $\pi$  is a vector space containing the space  $B^1(G, \pi)$  of coboundaries as linear subspace. The 1-*cohomology*  $H^1(G, \pi)$  is the quotient  $Z^1(G, \pi)/B^1(G, \pi)$ .

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The space  $B^1(G, \pi)$  is not necessarily closed in  $Z^1(G, \pi)$  (see Proposition 1) and the *reduced 1-cohomology* with coefficients in  $\pi$  is defined as  $\overline{H}^1(G, \pi) = Z^1(G, \pi) / \overline{B^1(G, \pi)}$ .

Assume now that  $G$  is compactly generated, that is,  $G = \cup_{n \in \mathbb{Z}} Q^n$  for a compact subset  $Q$ , which we can assume to be a neighbourhood of the identity  $e \in G$  and to be symmetric ( $Q^{-1} = Q$ ).

Harmonic 1-cocycles in  $Z^1(G, \pi)$ , with respect to an appropriate probability measure on  $G$ , form a set of representatives for the classes in the reduced cohomology  $\overline{H}^1(G, \pi)$ , as we will shortly explain. Such cocycles appear in [BeV] in the case where  $\pi$  is the regular representation of a discrete group  $G$ , in relation with the first  $\ell^2$ -Betti number of  $G$ ; they play an important role in Ozawa's recent proof of Gromov's polynomial growth theorem ([Oza]) as well as in the work [ErO] and [GoJ].

Harmonic 1-cocycles were implicitly introduced in [Gui, Theorem 2]; it was observed there that  $Z^1(G, \pi)$  can be identified with a closed subspace of the Hilbert space  $L^2(Q, \mathcal{H}, m_G)$ , where  $m_G$  is a (left) Haar measure on  $G$  and so  $\overline{H}^1(G, \pi)$  corresponds to the orthogonal complement  $B^1(G, \pi)^\perp$  of  $B^1(G, \pi)$  in  $Z^1(G, \pi)$ . Following [ErO], we prefer to embed  $Z^1(G, \pi)$  in a more general Hilbert space, defined by a class of appropriate probability measures similar to those appearing there. For this, we consider the word length on  $G$  associated to  $Q$ , that is, the map  $g \mapsto |g|_Q$ , where

$$|g|_Q = \min\{n \in \mathbb{N} : g \in Q^n\}.$$

**Definition 1.** A probability measure  $\mu$  on  $G$  is *cohomologically adapted* (or, more precisely, 1-cohomologically adapted) if it has the following properties:

- $\mu$  is symmetric;
- $\mu$  is absolutely continuous with respect to the Haar measure  $m_G$ ;
- $\mu$  is adapted: the support of  $\mu$  is a generating set for  $G$ ;
- $\mu$  has a second moment:  $\int_G |x|_Q^2 d\mu(x) < \infty$ .

Observe that the class of cohomologically adapted measures is independent of the generating compact set  $Q$ , since the length functions associated to two compact generating sets are bi-Lipschitz equivalent.

We consider the Hilbert space  $L^2(G, \mathcal{H}, \mu)$  of measurable square-integrable maps  $F : G \rightarrow \mathcal{H}$ . Then  $Z^1(G, \pi)$  is a subset of  $L^2(G, \mathcal{H}, \mu)$  (see Section 2). Moreover, the linear operator

$$\partial : \mathcal{H} \rightarrow Z^1(G, \pi), v \mapsto \partial_v$$

is bounded, has  $B^1(G, \pi)$  as range, and it is straightforward to check that its adjoint is  $-\frac{1}{2}M_\mu$ , where

$$M_\mu : Z^1(G, \pi) \rightarrow \mathcal{H}, \quad b \mapsto \int_G b(x) d\mu(x).$$

So, the orthogonal complement  $B^1(G, \pi)^\perp$  of  $B^1(G, \pi)$  in  $Z^1(G, \pi)$  can be identified with the space of harmonic cocycles in the sense of the following definition. In particular, the reduced cohomology  $\overline{H}^1(G, \pi)$  can be identified with  $\text{Har}_\mu(G, \pi)$ .

**Definition 2.** A cocycle  $b \in Z^1(G, \pi)$  is  $\mu$ -harmonic if  $M_\mu(b) = 0$ , that is,  $\int_G b(x) d\mu(x) = 0$ . We denote by  $\text{Har}_\mu(G, \pi)$  the space of  $\mu$ -harmonic cocycles in  $Z^1(G, \pi)$  and by

$$P_{\text{Har}} : L^2(G, \mathcal{H}, \mu) \rightarrow \text{Har}_\mu(G, \pi)$$

the orthogonal projection on  $\text{Har}_\mu(G, \pi)$ .

Observe that, by the cocycle relation,  $b \in Z^1(G, \pi)$  is  $\mu$ -harmonic if and only if it has the mean value property

$$b(g) = \int_G b(gx) d\mu(x) \quad \text{for all } g \in G.$$

In our opinion, the Hilbert space structure of  $\overline{H}^1(G, \pi)$  given by its realization as a space of harmonic cocycles, together with its module structure over the von Neumann algebra  $\pi(G)'$  (see below), deserves more attention than it has received so far in the literature. Our aim in this paper is to use this structure in relation with a natural notion of irreducibility for affine isometric actions (see Definition 3).

Our first result shows that harmonic 1-cocycles  $b$  are characterized by a remarkable minimality property of the space  $\overline{\text{span}(b(G))}$ , the closure of the linear span of  $b(G)$ .

**Theorem 1.** *Let  $G$  be a compactly generated group. Let  $(\pi, \mathcal{H})$  be an orthogonal or unitary representation of  $G$  and  $\mu$  a cohomologically adapted probability measure on  $G$ . Let  $b \in \text{Har}_\mu(G, \pi)$  be a  $\mu$ -harmonic cocycle. We have*

$$\overline{\text{span}(b(G))} = \bigcap_{b'} \overline{\text{span}(b'(G))},$$

where  $b'$  runs over the 1-cocycles in the cohomology class of  $b$  in  $\overline{H}^1(G, \pi)$ .

In particular, Theorem 1 shows that, for a  $\mu$ -harmonic cocycle  $b$ , the closed linear subspace spanned by  $b(G)$  only depends on the reduced cohomology class of  $b$  and not on the choice of  $\mu$ .

Recall that, given a cocycle  $b \in Z^1(G, \pi)$ , a continuous action  $\alpha_{\pi, b}$  of  $G$  on  $\mathcal{H}$  by affine isometries is defined by the formula

$$\alpha_{\pi, b}(g)v = \pi(g)v + b(g) \quad \text{for all } g \in G, v \in \mathcal{H}.$$

Conversely, let  $\alpha$  be a continuous action of  $G$  on  $\mathcal{H}$  by affine isometries. Denote by  $\pi(g)$  and  $b(g)$  the linear part and the translation part of  $\alpha(g)$  for  $g \in G$ . Then  $\pi$  is a unitary (or orthogonal) representation of  $G$  on  $\mathcal{H}$ ,  $b$  is a 1-cocycle in  $Z^1(G, \pi)$ , and  $\alpha = \alpha_{\pi, b}$ . For all this, see Chapter 2 in [BHV].

The following notion of irreducibility of affine actions was introduced in [Ner] and further studied in [BPV].

**Definition 3.** An affine isometric action  $\alpha$  of  $G$  on the complex or real Hilbert space  $\mathcal{H}$  is *irreducible* if  $\mathcal{H}$  has no non-empty, closed and proper  $\alpha(G)$ -invariant affine subspace.

First examples of irreducible affine isometric actions arise as actions  $\alpha_{\pi, b}$ , where  $\pi$  is an irreducible unitary representation of  $G$  with non trivial 1-cohomology and  $b \in Z^1(G, \pi)$  a cocycle which is not a coboundary. By [Sha1, Theorem 0.2], such a pair  $(\pi, b)$  always exists, provided  $G$  does not have Kazhdan's Property (T). A remarkable feature of irreducible affine isometric actions of a locally compact group  $G$  is that they remain irreducible under restriction to “most” lattices in  $G$  (see [Ner, 3.6], [BHV, Theorem 4.2]), whereas this is not true in general for irreducible unitary representations.

Let  $b \in Z^1(G, \pi)$ . Observe that  $\text{span}(b(G))$  is  $\alpha_{\pi, b}(G)$ -invariant. So, for  $\alpha_{\pi, b}$  to be irreducible, it is necessary that  $\text{span}(b(G))$  is dense in  $\mathcal{H}$ . This condition is not sufficient (see [BPV, Example 2.4]; however, see also Proposition 3 below). The following corollary of Theorem 1 relates harmonic cocycles to this question.

**Corollary 1.** *Let  $G, (\pi, \mathcal{H})$ , and  $\mu$  be as in Theorem 1. Let  $b \in Z^1(G, \pi)$  and  $P_{\text{Har}}b$  its projection on  $\text{Har}_{\mu}(G, \pi)$ .*

*(i) If  $\text{span}(P_{\text{Har}}b(G))$  is dense in  $\mathcal{H}$ , then the affine action  $\alpha_{\pi, b}$  is irreducible.*

*(ii) Assume that  $B^1(G, \pi)$  is closed; if the affine action  $\alpha_{\pi, b}$  is irreducible, then  $\text{span}(P_{\text{Har}}b(G))$  is dense in  $\mathcal{H}$ .*

**Remark 1.** (i) Point (ii) in Corollary 1 does not hold in general when  $B^1(G, \pi)$  is not closed; indeed, let  $G = \mathbb{F}_2$  denote the free group on 2 generators. Then  $H^1(G, \pi) \neq 0$  for every unitary representation  $\pi$  of  $G$  (see [Gui, §9, Example 1]). On the other hand, there exists an irreducible unitary representation  $\pi$  of  $G$  with  $\overline{H}^1(G, \pi) = 0$  (see [MaV, Theorem 1.1]), so that  $\text{Har}_{\mu}(G, \pi) = 0$  for any cohomologically adapted

probability measure  $\mu$  on  $G$ . Now, let  $b$  be a 1-cocycle in  $Z^1(G, \pi)$  which is not a coboundary. Then the affine action  $\alpha_{\pi, b}$  is irreducible.

(ii) Although we will not need it, we will give an explicit formula for the projection  $P_{\text{Har}} : Z^1(G, \pi) \rightarrow \text{Har}_\mu(G, \mu)$  in the case where  $B^1(G, \pi)$  is closed (see Proposition 4 below).

In view of Corollary 1, it is of interest to know when  $B^1(G, \pi)$  is closed. Write  $\mathcal{H} = \mathcal{H}^G \oplus \mathcal{H}^0$ , where  $\mathcal{H}^G$  is the space of  $\pi(G)$ -invariant vectors in  $\mathcal{H}$  and  $\mathcal{H}^0$  its orthogonal complement. Let  $\pi^0$  denote the restriction of  $\pi$  to  $\mathcal{H}^0$ . Observe that  $B^1(G, \pi^0) = B^1(G, \pi)$  and that  $Z^1(G, \pi^0)$  is closed in  $Z^1(G, \pi)$ ; so, the following result is both a (slight) strengthening and a consequence of Théorème 1 in [Gui].

**Proposition 1.** ([Gui]) *Let  $(\pi, \mathcal{H})$  be an orthogonal or unitary representation of the  $\sigma$ -compact group  $G$ . Then  $B^1(G, \pi)$  is closed in  $Z^1(G, \pi)$  if and only if  $(\pi^0, \mathcal{H}^0)$  does not weakly contain the trivial representation  $1_G$ .*

Our approach to the proof of Theorem 1 uses the fact, observed in [BPV, §3.1] and [BeV] that  $\overline{H}^1(G, \pi)$  is, in a natural way, a module over the (real or complex) von Neumann algebra  $\pi(G)'$ , which is the commutant of  $\pi(G)$  in  $\mathcal{B}(\mathcal{H})$ ; see Section 2. Viewing, as we do,  $\overline{H}^1(G, \pi)$  as the Hilbert space  $\text{Har}_\mu(G, \pi)$ , one is lead to the study of  $\text{Har}_\mu(G, \pi)$  as a Hilbert module over  $\pi(G)'$ .

For instance, if  $\mathcal{M} := \pi(G)'$  is a finite von Neumann algebra (that is, if there exists a faithful finite trace on  $\mathcal{M}$ ) then, we can define (as in [GHJ, Definition p.138] or [Bek, p. 327]) the *von Neumann dimension* of  $\overline{H}^1(G, \pi)$  as

$$\dim_{\mathcal{M}} \overline{H}^1(G, \pi) := \dim_{\mathcal{M}} \text{Har}_\mu(G, \pi) \in [0, +\infty) \cup \{+\infty\};$$

for more details, see Section 2. It is worth mentioning that in case  $\pi$  is the regular representation of a discrete group  $G$ ,  $\dim_{\mathcal{M}} \overline{H}^1(G, \pi)$  coincides with  $\beta_2^1(G)$ , the  $L^2$ -Betti number of  $G$  (see [BeV, Proposition 2]).

We now give some applications of von Neumann techniques to the problem of the existence of an irreducible affine isometric action of  $G$  with a given linear part  $\pi$ . First, using Corollary 1, we can reformulate Corollary 3.7 from [BPV] in our setting. Recall that a vector  $v$  in a Hilbert module over a von Neumann algebra  $\mathcal{M}$  is a *separating vector* for  $\mathcal{M}$  if  $Tv = 0$  for  $T \in \mathcal{M}$  implies  $T = 0$ .

**Proposition 2.** ([BPV])

(i) *Assume that  $\mathcal{M} = \pi(G)'$  has a separating vector  $b$  in  $\text{Har}_\mu(G, \pi)$ . Then  $\alpha_{\pi, b}$  is irreducible.*

(ii) Assume  $B^1(G, \pi)$  is closed and that  $\alpha_{\pi, b}$  is irreducible for some  $b \in \text{Har}_\mu(G, \pi)$ . Then  $b$  is a separating vector for  $\mathcal{M}$ .

For an application of the previous criterion in the case where  $G$  is a discrete finitely generated group and  $\pi$  a subrepresentation of a multiple of the regular representation of  $G$ , see [BPV, Theorem 4.25]. We extend this result to arbitrary factor representations, that is, to unitary representations  $(\pi, \mathcal{H})$  such that the von Neumann subalgebra  $\pi(G)''$  of  $\mathcal{B}(\mathcal{H})$  generated by  $\pi(G)$  is a factor (equivalently, such that  $\pi(G)'$  is a factor). Concerning general facts about factors, such as their type classification, see [Dix1].

**Theorem 2.** *Let  $(\pi, \mathcal{H})$  be a factor representation of the compactly generated locally compact group  $G$  on the separable complex Hilbert space  $\mathcal{H}$ . Assume that  $B^1(G, \pi)$  is closed in  $Z^1(G, \pi)$ . Set  $\mathcal{M} := \pi(G)'$  and let  $\mu$  be a cohomologically adapted probability measure on  $G$ . Depending on the type of  $\mathcal{M}$ , there exists  $b \in Z^1(G, \pi)$  such that  $\alpha_{\pi, b}$  is irreducible if and only if:*

- (i) *the factor  $\mathcal{M}$  is of type  $I_\infty$  or of type  $II_\infty$  and its commutant in  $\mathcal{B}(\text{Har}_\mu(G, \pi))$  is of infinite type (that is, of type  $I_\infty$  or  $II_\infty$ , respectively);*
- (ii) *the factor  $\mathcal{M}$  is of finite type (that is, of type  $I_n$  for  $n \in \mathbb{N}$  or of type  $II_1$ ) and  $\dim_{\mathcal{M}} \text{Har}_\mu(G, \pi) \geq 1$ ;*
- (iii) *the factor  $\mathcal{M}$  is of type  $III$  and  $\text{Har}_\mu(G, \pi) \neq \{0\}$ .*

**Remark 2.** Let  $(\pi, \mathcal{H})$  be a unitary representation of  $G$  such that  $B^1(G, \pi)$  is closed in  $Z^1(G, \pi)$ ; let

$$\pi = \int_{\Omega}^{\oplus} \pi_{\omega} d\nu(\omega)$$

be the central integral decomposition of  $\pi$ , so that the  $\pi_{\omega}$ 's are mutually disjoint factor representations of  $G$  (see [Dix2, Theorem 8.4.2]). One checks that one has a corresponding decomposition of  $\text{Har}_\mu(G, \pi)$  as a direct integral of Hilbert spaces:

$$\text{Har}_\mu(G, \pi) = \int_{\Omega}^{\oplus} \text{Har}_\mu(G, \pi_{\omega}) d\nu(\omega).$$

Moreover,  $B^1(G, \pi_{\omega})$  is closed in  $Z^1(G, \pi_{\omega})$  and there exists a separating vector for  $\pi(G)'$  in  $\text{Har}_\mu(G, \pi)$  if and only if there exists a separating vector for  $\pi_{\omega}(G)'$  in  $\text{Har}_\mu(G, \pi_{\omega})$  for  $\nu$ -almost every  $\omega$ . So, Theorem 2 can be used to check the existence of an irreducible affine with *any* unitary representation  $\pi$  as linear part (provided  $B^1(G, \pi)$  is closed in  $Z^1(G, \pi)$ ).

As an illustration of the use of Theorem 2, we will treat the example of a wreath product of the form  $\Gamma = G \wr \mathbb{Z}$  and a unitary representation  $\pi$  of  $\Gamma$  which factorizes through a representation of  $G$ ; the reduced cohomology of such groups was considered in [Sha2, §5.4].

**Theorem 3.** *Let  $G$  be a finitely generated group, and let  $(\pi, \mathcal{H})$  be a unitary representation of the wreath product  $\Gamma = G \wr \mathbb{Z}$  in the separable Hilbert space  $\mathcal{H}$ . Assume that  $\pi$  factorizes through  $G$  and that  $H^1(G, \pi) = 0$ .*

*(i) For a suitable cohomologically adapted probability measure  $\mu$  on  $\Gamma$ , the space  $\text{Har}_\mu(\Gamma, \mu)$  can be identified, as a module over  $\pi(\Gamma)'$ , with the Hilbert space  $\mathcal{H}$ .*

*(ii) There exists an irreducible affine action of  $\Gamma$  with linear part  $\pi$  if and only if the representation  $(\pi, \mathcal{H})$  is cyclic.*

*(iii) Assume that  $G$  is not virtually abelian (that is,  $G$  does not have an abelian normal subgroup of finite index). Then  $G$  has a factorial representation  $\pi$  for which  $\pi(G)'$  is of any possible type.*

**Remark 3.** (i) When  $\pi$  is a factor representation, a necessary and sufficient condition for the existence of a cyclic vector for  $\pi(G)$  (equivalently, a separating vector for  $\pi(G)'$ ) in  $\mathcal{H}$  is given in Theorem 2, with  $\mathcal{H}$  replacing  $\text{Har}_\mu(G, \mu)$  there.

(ii) By the Delorme-Guichardet theorem ([BHV, Theorem 2.12.4]), the condition  $H^1(G, \pi) = 0$  is satisfied for every unitary representation  $\pi$  of  $G$  if (and only if)  $G$  has Kazhdan's property (T).

## 2. THE SPACE OF HARMONIC COCYCLES AS A VON NEUMANN ALGEBRA MODULE

Let  $G$  be a locally compact group which is generated by a compact subset  $Q$ , which we assume to be a symmetric neighbourhood of the identity  $e \in G$ . Let  $(\pi, \mathcal{H})$  be an orthogonal or unitary representation of  $G$ . The map

$$b \mapsto \|b\|_Q = \sup_{x \in Q} \|b(x)\|$$

is a norm which generates the topology of uniform convergence on compact subsets and for which  $Z^1(G, \pi)$  is a Banach space.

Let  $\mathcal{M} := \pi(G)'$  be the commutant of  $\pi(G)$  in  $\mathcal{B}(\mathcal{H})$ , that is,

$$\mathcal{M} = \{T \in \mathcal{B}(\mathcal{H}) : T\pi(g) = \pi(g)T \text{ for all } g \in G\};$$

$\mathcal{M}$  is a (real or complex) von Neumann algebra, that is,  $\mathcal{M}$  is a unital self-adjoint subalgebra of  $\mathcal{B}(\mathcal{H})$  which is closed for the weak (or strong) operator topology.

As observed in [BPV, §3.1]),  $H^1(G, \pi)$  is a module over  $\mathcal{M}$ ; indeed, if  $b \in Z^1(G, \pi)$  and  $T \in \pi(G)'$ , then  $Tb \in Z^1(G, \pi)$ , where  $Tb$  is defined by

$$Tb(g) = T(b(g)) \quad \text{for all } g \in G;$$

moreover,  $T\partial_v = \partial_{Tv}$  for every vector  $v \in \mathcal{H}$ .

Let  $\mu$  be a cohomologically adapted probability measure on  $G$  (Definition 1). We consider the Hilbert space  $L^2(G, \mathcal{H}, \mu)$  of measurable mappings  $F : G \rightarrow \mathcal{H}$  such that

$$\|F\|_2^2 := \int_G \|F(x)\|^2 d\mu(x) < \infty.$$

Then every  $b \in Z^1(G, \pi)$  belongs to  $L^2(G, \mathcal{H}, \mu)$ ; indeed, the cocycle relation shows that

$$\|b(x)\| \leq |x|_Q \|b\|_Q \quad \text{for all } x \in G,$$

and hence

$$\|b\|_2^2 \leq \|b\|_Q^2 \int_G |x|_Q^2 d\mu(x) < \infty.$$

In fact, the norms  $\|\cdot\|_2$  and  $\|\cdot\|_Q$  on  $Z^1(G, \pi)$  are equivalent (see [ErO, Lemma 2.1]). So, we can (and will) identify  $Z^1(G, \pi)$  with a closed subspace of the Hilbert space  $L^2(G, \mathcal{H}, \mu)$ .

The von Neumann algebra  $\mathcal{M}$  acts on  $\mathcal{H}$  in the tautological way and on  $L^2(G, \mathcal{H}, \mu)$  by

$$TF(g) = T(F(g)) \quad \text{for all } T \in \pi(G)', F \in L^2(G, \mathcal{H}, \mu), g \in G,$$

preserving  $Z^1(G, \pi)$  and  $B^1(G, \pi)$ . Since the operator  $M_\mu : Z^1(G, \mu) \rightarrow \mathcal{H}$  is equivariant for these actions,  $\text{Har}_\mu(G, \pi) = \ker M_\mu$  as well as its orthogonal complement  $\overline{B^1(G, \pi)}$  are modules over  $\mathcal{M}$ .

The image of  $\mathcal{M}$  in  $\mathcal{B}(L^2(G, \mathcal{H}, \mu)) = \mathcal{B}(L^2(G, \mu)) \otimes \mathcal{H}$  is

$$\widetilde{\mathcal{M}} = I \otimes \pi(G)',$$

which is a von Neumann algebra isomorphic to  $\mathcal{M}$ . The orthogonal projection  $P_{\text{Har}} : L^2(G, \mathcal{H}, \mu) \rightarrow \text{Har}_\mu(G, \pi)$  belongs to the commutant

$$\widetilde{\mathcal{M}}' = \mathcal{B}(L^2(G, \mu)) \otimes \pi(G)''$$

of  $\mathcal{M}$  in  $\mathcal{B}(L^2(G, \mathcal{H}, \mu))$ , where  $\pi(G)''$  is the von Neumann algebra generated by  $\pi(G)$  in  $\mathcal{B}(\mathcal{H})$ . The commutant of  $\mathcal{M}$  in  $\text{Har}_\mu(G, \pi)$  is then the reduced von Neumann algebra (see Chap.1, §, Proposition 1 in [Dix1])

$$P_{\text{Har}} \widetilde{\mathcal{M}}' P_{\text{Har}} = P_{\text{Har}} (\mathcal{B}(L^2(G, \mu)) \otimes \pi(G)'') P_{\text{Har}}.$$

Assume now that  $\mathcal{M}$  is a finite von Neumann algebra, with faithful normalized trace  $\tau$ . Let  $L^2(\mathcal{M})$  be the Hilbert space obtained from



$\tau$  by the GNS construction. We identify  $\mathcal{M}$  with the subalgebra of  $\mathcal{B}(L^2(\mathcal{M}))$  of operators given by left multiplication with elements from  $\mathcal{M}$ . The commutant of  $\mathcal{M}$  in  $\mathcal{B}(L^2(\mathcal{M}))$  is  $\mathcal{M}' = J\mathcal{M}J$ , where  $J : L^2(\mathcal{M}) \rightarrow L^2(\mathcal{M})$  is the conjugate linear isometry which extends the map  $\mathcal{M} \rightarrow \mathcal{M}, x \mapsto x^*$ . The trace on  $\mathcal{M}'$ , again denoted by  $\tau$ , is defined by  $\tau(JxJ) = \tau(x)$  for  $x \in \mathcal{M}$ .

The  $\mathcal{M}$ -module  $L^2(G, \mathcal{H}, \mu)$  can be identified with an  $\mathcal{M}$ -submodule of  $L^2(\mathcal{M}) \otimes \ell^2$ , with  $\mathcal{M}$  acting on  $L^2(\mathcal{M}) \otimes \ell^2$  by  $T \mapsto T \otimes I$ . The orthogonal projection  $Q : L^2(\mathcal{M}) \otimes \ell^2 \rightarrow L^2(G, \mathcal{H}, \mu)$  belongs to the commutant of  $\mathcal{M}$  in  $\mathcal{B}(L^2(\mathcal{M}) \otimes \ell^2)$ , which is  $\mathcal{M}' \otimes \mathcal{B}(\ell^2)$ . The projection  $P = P_{\text{Har}} \circ Q$  belongs therefore to the commutant of  $\mathcal{M}$  in  $\mathcal{B}(L^2(\mathcal{M}) \otimes \ell^2)$ .

Let  $\{e_n\}_n$  be a basis of  $\ell^2$  and let  $(P_{ij})_{i,j}$  be the matrix of  $P$  with respect to the decomposition  $L^2(\mathcal{M}) \otimes \ell^2 = \oplus_i (L^2(\mathcal{M}) \otimes e_i)$ . Then every  $P_{ij}$  belongs to  $\mathcal{M}'$  and the von Neumann dimension of the  $\mathcal{M}$ -module  $\text{Har}_\mu(G, \pi)$  is

$$\dim_{\mathcal{M}} \mathcal{H} = \sum_i \tau(P_{ii}).$$

### 3. PROOFS OF THE MAIN RESULTS

**3.1. Proof of Theorem 1.** Let  $b_0 \in \text{Har}_\mu(G, \pi)$ . Let  $b_1 \in \overline{B^1(G, \pi)}$  and set  $b := b_0 + b_1$ . We claim that  $b_0(G)$  is contained in the closure of  $\text{span}(b(G))$ .

Indeed, let  $\mathcal{K}$  denote the closure of  $\text{span}(b(G))$  and  $P_{\mathcal{K}} : \mathcal{H} \rightarrow \mathcal{K}$  the corresponding orthogonal projection. Since  $\mathcal{K}$  is  $\pi(G)$ -invariant,  $P_{\mathcal{K}}$  belongs to the commutant  $\pi(G)'$  of  $\pi(G)$ . Therefore (see Section 2),  $P_{\mathcal{K}}b_0$  is contained in  $\text{Har}_\mu(G, \pi)$  and  $P_{\mathcal{K}}b_1$  is contained in  $\overline{B^1(G, \pi)}$ . On the other hand, since  $b$  take its values in  $\mathcal{K}$ , we have that

$$P_{\mathcal{K}}b = b = b_0 + b_1.$$

It follows that  $P_{\mathcal{K}}b_0 = b_0$  and  $P_{\mathcal{K}}b_1 = b_1$ . Therefore,

$$b_0(G) \subset \mathcal{K} = \overline{\text{span}(b(G))},$$

as claimed. ■

### 3.2. A characterization of irreducible affine isometric actions.

We will need for the proof of Corollary 1 one of the several characterizations of irreducible affine actions from Proposition 2.1 in [BPV]; for the convenience of the reader, we give a direct and short argument.

**Proposition 3.** ([BPV]) *For  $b \in Z^1(G, \pi)$ , the following properties are equivalent:*

(i) *the action  $\alpha = \alpha_{\pi, b}$  is irreducible;*

(ii) the linear span of  $(b + \partial_v)(G)$  is dense in  $\mathcal{H}$  for every  $v \in \mathcal{H}$ .

**Proof** Observe that

$$\alpha_{\pi, b + \partial_v}(g) = t_{-v} \circ \alpha_{\pi, b}(g) \circ t_v \quad \text{for all } g \in G, v \in \mathcal{H},$$

where  $t_v$  is the translation by  $v$ . So,  $\alpha_{\pi, b}$  is irreducible if and only if  $\alpha_{\pi, b + \partial_v}$  is irreducible. This shows that (i) implies (ii).

To show the converse implication, let  $F$  be a non empty closed  $\alpha_{\pi, b}(G)$ -invariant affine subspace of  $\mathcal{H}$ . Then  $F = v + \mathcal{K}$  for a vector  $v \in \mathcal{H}$  and a closed linear subspace  $\mathcal{K}$  of  $\mathcal{H}$ . Set  $b_0 := b + \partial_v$ . Then

$$v + b_0(g) = \alpha_{\pi, b}(g)v \in F \quad \text{for all } g \in G,$$

and  $b_0(G)$  is hence contained in  $\mathcal{K}$ . Therefore,  $\mathcal{K} = \mathcal{H}$ , since  $\text{span}(b_0(G))$  is dense in  $\mathcal{H}$ . ■

**3.3. Proof of Corollary 1.** Let  $b \in Z^1(G, \pi)$  and set  $b_0 := P_{\text{Har}} b \in \text{Har}_\mu(G, \pi)$ .

(i) Assume that  $\text{span}(b_0(G))$  is dense in  $\mathcal{H}$ . By Theorem 1, the linear span of  $(b + \partial_v)(G)$  is dense for every  $v \in \mathcal{H}$ , and Proposition 3 shows that  $\alpha_{\pi, b}$  is irreducible.

(ii) Assume now that  $B^1(G, \pi)$  is closed in  $Z^1(G, \pi)$  and that  $\alpha_{\pi, b}$  is irreducible. Write  $b = b_0 + \partial_{v_0}$  for  $b_0 = P_{\text{Har}} b$  and  $v_0 \in \mathcal{H}$ . Then  $\alpha_{\pi, b_0} = \alpha_{\pi, b - \partial_{v_0}}$  is also irreducible, by Proposition 3; hence,  $\text{span}(b_0(G))$  is dense. ■

**3.4. Proof of Theorem 2.** Let  $(\pi, \mathcal{H})$  be a unitary representation of  $G$ ; we assume that  $B^1(G, \pi)$  is closed in  $Z^1(G, \pi)$ . Let  $\mu$  be a cohomologically adapted probability measure on  $G$ .

In view of Proposition 2, we have to investigate under which conditions  $\mathcal{M} = \pi(G)'$  has a separating vector in  $\text{Har}_\mu(G, \pi)$ . We may assume that  $\text{Har}_\mu(G, \pi) \neq \{0\}$ .

Observe that a vector in  $\text{Har}_\mu(G, \pi)$  is separating for  $\mathcal{M}$  if and only if it is cyclic for the commutant  $\mathcal{N}$  of  $\mathcal{M}$  in  $\mathcal{B}(\text{Har}_\mu(G, \pi))$ . Three cases can occur.

- *First case:*  $\mathcal{N}$  is an infinite factor. Then  $\mathcal{M}$  always has a separating vector (see Corollaire 11 in Chap. III, §8 of [Dix1]).
- *Second case:*  $\mathcal{N}$  is a finite factor and  $\mathcal{M}$  is an infinite factor. Then  $\mathcal{N}$  has a cyclic vector in  $\text{Har}_\mu(G, \pi)$  if and only if  $\dim_{\mathcal{N}} \text{Har}_\mu(G, \pi) \leq 1$  (see [Bek, Corollary 1]). For this to happen a necessary condition is that  $\mathcal{M}$  is a finite factor. So,  $\mathcal{M}$  has no separating vector.

- *Third case:*  $\mathcal{N}$  and  $\mathcal{M}$  are finite factors. In this case, we have (see [GHJ, Prop. 3.2.5])

$$\dim_{\mathcal{M}} \text{Har}_{\mu}(G, \pi) \dim_{\mathcal{N}} \text{Har}_{\mu}(G, \pi) = 1;$$

hence,  $\mathcal{M}$  has a separating vector in  $\text{Har}_{\mu}(G, \pi)$  if and only if

$$\dim_{\mathcal{M}} \text{Har}_{\mu}(G, \pi) \geq 1.$$

Claims (i), (ii), and (iii) follow from this discussion. ■

**3.5. Proof of Theorem 3.** We first consider the general case of the wreath product  $\Gamma = G \wr H$  of two finitely generated groups  $G$  and  $H$ . Recall that  $\Gamma = G \ltimes H^{(G)}$ , for  $H^{(G)} = \bigoplus_{g \in G} H$  and  $G$  acts on  $H^{(G)}$  by shifting the copies of  $H$ . We view  $H$  as a subgroup of  $\Gamma$ , by identifying it with the copy of  $H$  inside  $H^{(G)}$  indexed by  $e$ .

Let  $S_1$  and  $S_2$  finite symmetric generating sets for  $G$  and  $H$ , respectively. Then  $S_1 \cup S_2$  is a finite symmetric generating set for  $\Gamma$ . Let  $\mu_1$  and  $\mu_2$  be cohomologically adapted probability measures on  $G$  and  $H$  respectively. Then  $\mu = \frac{1}{2}(\mu_1 + \mu_2)$  is a cohomologically adapted probability measure on  $\Gamma$ .

Let  $(\pi, \mathcal{H})$  be a unitary representation of  $G$ , viewed as representation of  $\Gamma$ . We have orthogonal  $\pi(\Gamma)$ -invariant decompositions

$$\ell^2(\Gamma, \mathcal{H}, \mu) = \ell^2(G, \mathcal{H}, \mu_1) \oplus \ell^2(H, \mathcal{H}, \mu_2)$$

and

$$\text{Har}_{\mu}(\Gamma, \pi) = \text{Har}_{\mu_1}(G, \pi) \oplus \text{Har}_{\mu_2}(H, \pi).$$

Since  $\pi$  is trivial on  $H$ , the space  $Z^1(H, \pi)$  coincides with the set  $\text{Hom}(H, \mathcal{H})$  of homomorphisms  $H \rightarrow \mathcal{H}$ . Observe that every  $b \in \text{Hom}(H, \mathcal{H})$  is  $\mu_2$ -harmonic, since

$$\sum_{h \in H} b(h) \mu_2(h) = \sum_{h \in H} b(-h) \mu_2(h) = - \sum_{h \in H} b(h) \mu_2(h).$$

Hence,  $\text{Har}_{\mu_2}(H, \pi) = \text{Hom}(H, \mathcal{H})$  (alternatively, this follows from the fact that  $B^1(H, \pi) = B^1(H, 1_H)$  is trivial); therefore, we have

$$\text{Har}_{\mu}(\Gamma, \pi) = \text{Har}_{\mu_1}(G, \pi) \oplus \text{Hom}(H, \mathcal{H}).$$

We specialize by taking  $H = \mathbb{Z}$ ; then  $\text{Hom}(H, \mathcal{H})$  can be identified with  $\mathcal{H}$  and we have

$$\text{Har}_{\mu}(\Gamma, \pi) = \text{Har}_{\mu_1}(G, \pi) \oplus \mathcal{H};$$

moreover, the action of the von Neumann algebra  $\pi(\Gamma)' = \pi(G)'$  on  $\text{Har}_{\mu}(G, \mu)$  corresponds to the direct sum of the actions of  $\pi(G)'$  on  $\text{Har}_{\mu_1}(G, \mu_1)$  and on  $\mathcal{H}$ .

In particular, when the 1-cohomology  $H^1(G, \pi)$  is trivial, we have

$$\text{Har}_\mu(\Gamma, \pi) = \mathcal{H},$$

so that Claim (i) is proved. Claim (ii) follows from Proposition 2.

To show Claim (iii), assume that  $G$  is not virtually abelian. Then  $G$  is not of type  $I$ , by Thoma's theorem ([Tho, Satz 6]).

First, observe that  $G$  has an irreducible unitary representation  $\sigma$  of infinite dimension; indeed, otherwise,  $G$  would be a liminal (or CCR) group and hence of type  $I$ , by [Dix2, 13.9.7]. Set  $\pi = n\sigma$ , a multiple of  $\sigma$  for  $n \in \mathbb{N}$  or  $n = \infty$ ; then  $\pi(G)'$  is of type  $I_n$ .

Next, since  $G$  is not of type  $I$ ,  $G$  has a factorial representation  $\pi$  such that both  $\pi(G)''$  and  $\pi(G)'$  are of type  $II_1$ , by [Tho, Lemma 19]. Then  $\rho := \infty\pi$  is factorial and  $\rho(G)'$  is of type  $II_\infty$ .

Finally,  $G$  has a factor representation such that  $\pi(G)''$  (and hence  $\pi(G)'$ ) is of type  $III$ , by Glimm's theorem [Gli, Theorem 1]). ■

#### 4. AN EXPLICIT FORMULA FOR THE PROJECTION ON HARMONIC COCYCLES

We give an explicit formula for the orthogonal projection  $P_{\text{Har}}$  in terms of an averaging (or Markov) operator associated to  $\mu$ , in the case where  $B^1(G, \pi)$  is closed.

Consider the operator  $\pi^0(\mu) \in \mathcal{B}(\mathcal{H}^0)$  defined by

$$\pi^0(\mu)v = \int_G \pi(x)v d\mu(x) \quad \text{for all } v \in \mathcal{H}^0.$$

The operator  $\pi^0(\mu) - I : \mathcal{H}^0 \rightarrow \mathcal{H}^0$  is invertible if and only if  $\pi^0$  does not weakly contain the trivial representation  $1_G$  (see Proposition G.4.2 in [BHV]); in view of Proposition 1, this is the case if and only if  $B^1(G, \pi)$  is closed.

**Proposition 4.** *Assume that  $B^1(G, \pi)$  is closed. For  $b \in Z^1(G, \pi)$ , we have  $P_{\text{Har}}b = b - \partial_v$ , where*

$$v = (\pi^0(\mu) - I)^{-1}(M_\mu(b)).$$

**Proof** Indeed, observe first that  $M_\mu(b) \in \mathcal{H}^0$ ; indeed, for every  $w \in \mathcal{H}^G$ , we have

$$\begin{aligned} \langle M_\mu(b), w \rangle &= \int_G \langle b(x), w \rangle d\mu(x) = \int_G \langle b(x), \pi(x)w \rangle d\mu(x) \\ &= \int_G \langle \pi(x^{-1})b(x), w \rangle d\mu(x) = - \int_G \langle b(x^{-1}), w \rangle d\mu(x) \\ &= - \int_G \langle b(x), w \rangle d\mu(x) = - \langle M_\mu(b), w \rangle. \end{aligned}$$

Moreover, for  $v = (\pi^0(\mu) - I)^{-1}(M_\mu(b))$ , we have

$$M_\mu(\partial_v) = \int_G (\pi(x)v - v)d\mu(x) = (\pi^0(\mu) - I)v = M_\mu(b). \blacksquare$$

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